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Some Results on Ulam Hyers Stability of Integrodifferential Equations with nonlocal Condition on infinite interval

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Abstract:-

In this paper, Pachpatte's inequality is used to discuss the Ulam Hyers stabilities for Volterra integrodifferential equations with nonlocal condition in Banach spaces on infinite interval. Example is given to show the applicability of our obtained result.

Keywords: Ulam Hyers stability; Ulam Hyers Rassias stability; Integral inequality.
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1) Introduction

The Mathematician Ulam had developed the stability problem pertaining to functional equations (see [[6],[7]]). The Ulam problem was stated as Under what conditions there exist an additive mapping near an approximately additive mapping. Initially Hyers [9] tried to find answer to the question of Ulam (for the additive mapping) in the case of Banach spaces. Thereafter, Rassias [11] extended Ulam–Hyers stability concept by introducing new function variables. In the literature, these concepts of stabilities are known asUlam stability, Ulam Hyers stability and Ulam Hyers Rassias stability. The basic Ulam stability problem of functional equations has been extended to different types of equations. It is observed that the Ulam stability theory plays an important role in the study of differential equations, integral equations, difference equations, fractional differential equations etc. For any kind of equations, Ulam stability problem is about (see [8, 10]) When should the solutions of an equation, differing slightly from a given one, must be close to a solution of the given equation? At the end of 19th and 20th century, a large number of researchers contributed to the stability idea of Ulam's type for various types of differential and Integral equations. There are many applications of Ulam's type stability in tackling problems related to Operation reserarch ,numerical analysis, control theory, and many more,in such situations to get an exact solution is very difficult.
For basic development on Ulam stability, see[2,7,8,9,11,12,13,14,15].Rus[15] studied Ulam–Hyers stabilities for the first order Differential Equation$x'(t) = f(t, x(t)$.
Using the fixed point technique, Jung [10] proved Ulam–Hyersstability for$x'(t) = \int_c^t f(t, x(s))ds$ where c is fixed real number.

The notion of 'nonlocal' condition has been introduced to extend the study of the classical initial value problems. It is more precise for describing nature phenomena than the classical condition since more information is taken into account.see[4-5]The study of abstract nonlocal semilinear initial-value problems was initiated by L. Byszewski. We motivated by work of K.D.Kucche, P.U. Shikhare [2].

The purpose of this paper is to study Ulam stability problem of functional equations with nonlocal Condition on infinite interval of the form:

$$x'(t) = A(t)x(t) + f(t, x(t), \int_0^t g(t, s, x(s))ds), \quad t \in J = [0, \infty] \tag{1.1}$$

$$x(0) + H(x) = x_0 \tag{1.2}$$

where A is an infintesimal generator of strongly continuous semigroup of bounded linear operator T(t) in X with domain $D(A)$, the unknown $x(\cdot)$ takes values in the Banach space X; $f: J \times X \times X \to X$, $g: C(J \times J, X) \to X$,$H: C(J \times J, X) \to X$ are appropriate continuous functions and x_0 is given element of X.

The paper is organized as follows: We discussed the preliminaries. We dealt with study of Ulam Hyers Rassias stability of VIE with nonlocal condition in Banach space. Finally we gave the example to illustrate the application of our result.

2) Preliminaries

In this section, we recall some necessary definitions and theorems which will be used in the sequel see Pazy [1] and Pachpatte[3]

Definition:- A one parameter family $T(t)_{t \geqq 0}$ of bounded linear operators from Banach space X into X is called strongly continuous semigroup (or C_0- semigroup) of operators on X if

- $T(0) = I$ the identity operator ,
- $T(t + s) = T(t)T(s) = T(s)T(t), \quad t, s \geq 0,$
- $\lim_{t \to 0} T(t)x = x \; \forall x \in X$

Definition:-The infinitesimal generator of the C_0 semigroup $T(t)_{t \geqq 0}$ is the linear operator $A: D(A) \subseteq X \to X$ defined by

$$Ax = \lim_{t \to 0^+} \frac{T(t)x - x}{t}, \text{for every } x \in D(A)$$

where

$$D(A) = \{x \in X: \lim_{t \to 0^+} \frac{T(t)x - x}{t} \text{exist in X} \}$$

Theorem 2.1 ([1])Let $T(t)_{t \geqq 0}$ is a C_0 semigroup There exist constant $\omega \geq 0$ and $M \geq 1$ such that $\| T(t) \| \leq Me^{\omega t}$,$0 \leq t < \infty$

Pachpatte's inequality given below plays crucial role in our further analysis.

Theorem2.2 ([[3], p. 39]). Let $u(t)$, $f(t)$ and $q(t)$ be nonnegative continuous functions defined on $\mathbb{R}_+$, and $n(t)$ be a positive and nondecreasing continuous function defined on $\mathbb{R}_+$ for which the inequality

$$u(t) \leq n(t) + \int_0^t f(s)[u(s) + \int_0^s q(\tau)u(\tau)d\tau]ds$$

hold for $t \in \mathbb{R}_+$ .Then

$$u(t) \leq n(t)[1 + \int_0^t f(s)\exp(\int_0^s [f(\tau) + q(\tau)]d\tau)\text{ds}]$$

for $t \in \mathbb{R}_+$

3) Ulam Hyers stabilities of Semilinear VIE and its application

In this section, we establish Ulam Hyers stabilities of similinear VIE

$$x'(t) = Ax(t) + f\left(t, x(t), \int_0^t g\big(t, s, x(s)\big)ds\right), \quad t \in J \tag{3.1}$$

$$x(0) + H(x) = x_0, \tag{3.2}$$

in a Banach Space $(X, \|.\|)$ where

1. $J = [0, \infty]$
2. $A: X \to X$ is an infinitesimal generator of C_0-semigroup $T(t)_{t \geqq 0}$ in X;
3. $f: J \times X \times X \to X$ and $g: J \times J \times X \to X$,$H: C(J \times X) \to X$ are continuous functions.

Definition 3.1 Let $\mathrm{T(t)}_{t \geqq 0}$ is a $\mathrm{C_0}$-semigroup of bounded linear operators in X with infinitesimal generator A and $\mathrm{f} \in \mathrm{L^1(J, X)}$. A function $\mathrm{x} \in \mathrm{C(J, X)}$ given by

$$x(t) = T(t)[x_0 - H(x)] + \int_0^t T(t-s) f(s, x(s), \int_0^s g(s, \tau, x(\tau)) d\tau) ds,$$

is called the mild solution of initial value problem.

$$x'(t) = Ax(\mathrm{t}) + f(t, x(t), \int_0^t g(t, s, x(s)) ds)$$
$$x(0) + H(x) = x_0 \qquad (3.3)$$

Definition 3.2 Equation (3.1)-(3.2) is Ulam Hyers stable if there exists a real number $\mathrm{C_f} > 0$ such that for each $\varepsilon > 0$ and for each solution $\mathrm{y} \in \mathrm{C'(J, X)}$ of the inequation The function $\mathrm{x} \in \mathrm{B}$ satisfies the integral equation

$$\| y'(t) - Ay(t) - f(t, y(t), \int_0^t g(t, s, y(s)) ds) \| \leq \varepsilon, t \in J \qquad (3.4)$$

$\exists$ a mild solution $x: J \to X$ in $C(J, X)$ of (3.1)-(3.2) with

$$\| y(t) - x(t) \| \leq C_f \varepsilon, \quad t \in J \qquad (3.5)$$

Definition 3.3 Equation (3.1)-(3.2) is Ulam Hyers Rassias stable, with respect to the positive non-decreasing continuous function $\psi: \mathrm{J} \in \mathbb{R}_+$,if there exists $\mathrm{C_{f,\psi}} > 0$ such that for each $\varepsilon > 0$ and for each solution $\mathrm{y} \in \mathrm{C_1(J, X)}$ of the inequation

$$\| y'(t) - Ay(t) - f\left(t, y(t), \int_0^t g\big(t, s, y(s)\big) ds\right) \| \leq \varepsilon \psi(t), \quad t \in J \qquad (3.6)$$

there exists a mild solution $x: J \to X$ in $C(J, X)$ of (3.1)-(3.2) with

$$\| y(t) - x(t) \| \leq C_{f,\psi} \varepsilon \psi(t), \qquad t \in J.$$

Definition 3.4 Equation (3.1)-(3.2) is generalized Ulam Hyers Rassias stable, with respect to the positive non-decreasing continuous function $\psi: \mathrm{J} \in \mathbb{R}_+$,if there exists $\mathrm{C_{f,\psi}} > 0$ such that for each solution $\mathrm{y} \in \mathrm{C_1(J, X)}$ of the inequation

$$\| y'(t) - Ay(t) - f\left(t, y(t), \int_0^t g\big(t, s, y(s)\big) ds\right) \| \leq \psi(t), \quad t \in J \qquad (3.7)$$

there exists a mild solution $x: J \to X$ in $C(J, X)$ of (3.1)-(3.2) with

$$\| y(t) - x(t) \| \leq C_{f,\phi} \psi(t), \qquad t \in J \qquad (3.8)$$

Remark 3.1

A function $y \in C^1(J, X)$ is a solution of in equation (3.4) if there exists a function $h \in C(J, X)$ (which depends on y) such that

1. $\| h(t) \| \leq \varepsilon, t \in J$.
2. $y'(t) = Ay(t) + f(t, y(t), \int_0^t g(t, s, y(s)) ds) + h(t), t \in J$.

Remark 3.2

If $y \in C^1(J, X)$ satisfies inequation (3.4) then y is a solution of the following integral inequation:

$$\| y(t) - T(t)[y_0 - H(y)] - \int_0^t T(t-s) f(s, y(s), \int_0^s g(s, \tau, y(\tau)) d\tau) ds \|$$

$\leq \varepsilon \int_0^t \parallel T(t-s) \parallel ds \quad with\ t \in J$(3.9)

indeed ,if $y \in C'(J,X)$ satisfies inequation (3.4) by Remark 3.1, we have

$$y'(t) = Ay(t) + f(t, y(t), \int_0^t g(t,s,y(s))ds) + h(t), t \in J \quad (3.10)$$

This implies that

$$y(t) = T(t)[y_0 - H(y)] + \int_0^t T(t-s)[f(s,y(s), \int_0^s g(s,\tau,y(\tau))d\tau) + h(s)]ds \quad (3.11)$$

Therefore $\parallel y(t) - T(t)[y_0 - H(y)] + \int_0^t T(t-s)[f(s,y(s), \int_0^s g(s,\tau,y(\tau))d\tau)]ds \parallel$

$$\leq \int_0^t \parallel T(t-s) \parallel\parallel h(s) \parallel ds \quad (3.12)$$

$$\leq \varepsilon \int_0^t \parallel T(t-s) \parallel ds \quad (3.13)$$

We list the following hypotheseson f and g, H for our convenience:

$(\boldsymbol{H1})'$ (i) Let $f \in C([0,\infty) \times X \times X, X)$ and there exists nonnegative continuous unction $L(\cdot) \in L^1[0,\infty)$such that

$\parallel f(t,x_1,x_2) - f(t,y_1,y_2) \parallel \leq L(t)(\parallel x_1 - y_1 \parallel + \parallel x_2 - y_2 \parallel)$.

(ii) Let$g \in C([0,+\infty) \times [0,+\infty) \times X, X)$and there existsnonnegative continuous function $G(\cdot) \in L^1[0,+\infty)$ such that

$\parallel g(t,s,x_1) - g(t,s,y_1) \parallel \leq G(t) \parallel x_1 - y_1 \parallel$, for all $t, s \in I, x_i, y_i \in X$ $(i = 1,2)$.

(iii) For positive, non-decreasing continuous and bounded function $H: C(J \times J, X) \to X$ there exist positive constant $K_1 \in \mathbb{R}$ such that $\parallel H(x) - H(y) \parallel \leq K_1 \parallel x - y \parallel$for every $x, y \in X$.

$(\boldsymbol{H2})'$ The function $\psi : [0,+\infty) \to \mathbb{R}_+$is positive, non-decreasing and continuous and there exists $\lambda > 0$ such that$\int_0^t \parallel T(t-s) \parallel \psi(s)ds \leq \lambda\psi(t)$,$t \in [0,+\infty)$

4) Main Result

In this section ,we prove the fundamental result.

Theorem:-Let f and g in (3.1) satisfy the hypothesis (H1)′ and assume(H2)′ that hold. Then Eq. (3.1) is Ulam–Hyers–Rassias stable with respect to ψ

Proof. *Let*$\psi : [0,+\infty) \to \mathbb{R}_+$ be a positive, non-decreasing and continuousfunction. Let $y \in C^1([0,+\infty), X)$satisfies inequation

$$\|y'(t) - Ay(t) - f\left(t, y(t), \int_0^t g\big(t,s,y(s)\big)\, ds\right)\| \leq \epsilon\psi(t), \forall\ t \in [0,+\infty)$$

Then using$(H2)'$ and proceeding as in Remark 3.2, we obtain

$$\|y(t) - T(t)[y_0 - H(y)] - \int_0^t T(t-s) f\left(s, y(s), \int_0^s g\big(s,\tau,y(\tau)\big)d\tau\right) ds\|$$

$\leq \epsilon\lambda\psi(t), \forall\ t \in [0,+\infty)$ (4.1)

If $x \in C([0,+\infty), X)$be the mild solution of the problem

$$x'(t) = Ax(t) + f\left(t, x(t), \int_0^t g\big(t,s,x(s)\big)ds\right), \forall t \in [0,+\infty),$$

$$x(0) = y(0).$$

$$x_0 - H(x) = y_0 - H(y)$$

Then we have

$$x(t) = T(t)[x_0 - H(x)] + \int_0^t T(t-s) f\left(s, x(s), \int_0^s g\big(s, \tau, x(\tau)\big) d\tau\right) ds$$

$$x(t) = T(t)[y_0 - H(y)] + \int_0^t T(t-s) f\big(s, x(s), \int_0^s g\big(s, \tau, x(\tau)\big) d\tau\big) ds \quad (4.2)$$

Using Eq. (4.2), inequation (4.1) and the hypothesis $(H1)'$, we get

$$\|y(t) - x(t)\| \le \|y(t) - T(t)[y_0 - H(y)] - \int_0^t T(t-s) f\left(s, x(s), \int_0^s g\big(s, \tau, x(\tau)\big) d\tau\right) ds$$

$$\le \|y(t) - T(t)[y_0 - H(y)] - \int_0^t T(t-s) f\left(s, x(s), \int_0^s g\big(s, \tau, x(\tau)\big) d\tau\right) ds$$

$$+ \int_0^t T(t-s) f\left(s, y(s), \int_0^s g\big(s, \tau, y(\tau)\big) d\tau\right) ds$$

$$- \int_0^t T(t-s) f\left(s, y(s), \int_0^s g\big(s, \tau, y(\tau)\big) d\tau\right) ds$$

In above step we add and substract one term.Using(4.1) we get.

$$\|y(t) - x(t)\| \le \epsilon\lambda\psi(t) + \int_0^t M e^{\omega(t-s)} L(s) \times \left(\|y(s) - x(s)\| + \int_0^s G(\tau)[\|y(\tau) - x(\tau)\|] d\tau\right) ds$$

Applying Pachpatte's inequality given in Theorem (2.2)to above inequality with $u(t) = \|y(t) - x(t)\|$,$n(t) = \epsilon\lambda\psi(t), f(s) = Me^{\omega(t-s)}L(s), q(\tau) = \mathrm{G}(\tau)$ holds for $t \in [0, +\infty)$ Then

$$\|y(t) - x(t)\| \le \epsilon\lambda\psi(t)\left[1 + \int_0^t ML(s)e^{\omega(t-s)} exp\left(\int_0^s \big[ML(\tau)e^{\omega(t-\tau)} + G(\tau)\big] d\tau\right) ds\right]$$

$$\le \epsilon\lambda\psi(t)\left[1 + \int_0^t ML(s)e^{\omega(t-s)} exp\left(\int_0^\infty \big[ML(\tau)e^{\omega(t-\tau)} + G(\tau)\big] d\tau\right) ds\right]$$

By putting $C_{f,\psi} = \lambda\left[1 + \int_0^t ML(s)e^{\omega(t-s)} exp\big(\int_0^\infty \big[ML(\tau)e^{\omega(t-\tau)} + G(\tau)\big] d\tau\big) ds\right]$

We obtain $\|y(t) - x(t)\| \le \epsilon\psi(t)C_{f,\psi}$,$\forall\, t \in [0, +\infty)$

This proves Eq. (3.1) is Ulam–Hyers–Rassias stable with respect to the function ψ.

The proof ends. □

Corollary :- Let f and g in (3.1) satisfy the condition $(\mathrm{H1})'$ and assume that $(\mathrm{H2})'$hold. Then Eq. (3.1) is generalized Ulam–Hyers–Rassias stable withrespect to ψ on $\mathrm{J} = [0, +\infty)$.

Proof. Taking$\epsilon = 1$ in the proof of Main Theorem , we obtain.

$$\|y(t) - x(t)\| \le C_{f,\psi}\psi(t), \forall\, t \in [0, +\infty)$$

Therefore, (3.1) is generalized Ulam–Hyers–Rassias stable with respect to the function ψ on $J = [0, +\infty)$.

Remark 3.3:- Equation (3.1) is not Ulam–Hyers stable on the interval $J = [0, +\infty)$.

5) **Application**

We provide the following example in support of theargument we made in Remark 3.3.

Example Consider the initial value problem for VIE in the Banach space $(\mathbb{R}, |\cdot|)$ with infinitesimal generator $A = 0$:

$$x'(t) = \frac{2}{5} - \frac{45}{7}\cos(x(t)) - \frac{5}{7}\sin\left(3(x(t))\right) + \frac{24}{7}\int_0^t sin^3(x(s))ds,$$

$\forall\, t \in [0, +\infty)$.(5.1)

$x(0) + x(e^{i\pi} + 1) = 0$ (5.2)

Therefore,in this example we have

$A = 0$,

$$f\left(t,x(t), \int_0^t g(t,s,x(s))ds\right) = \frac{2}{5} - \frac{45}{7}\cos(x(t)) - \frac{5}{7}\sin\left(3(x(t))\right) + \frac{24}{7}\int_0^t sin^3(x(s))ds,$$

$$g(t,s,x(s)) = \frac{24}{7} sin^3(x(s)),$$

And $H(x) = x(e^{i\pi} + 1)$

Where domains of functions as given below

$f : [0, +\infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, g : [0, +\infty) \times [0, +\infty) \times \mathbb{R} \to \mathbb{R}., H: C([0, +\infty) \times \mathbb{R}) \to \mathbb{R}$

Then the initial value problem (5.1)–(5.2) takes the form

$$x'(t) = f\left(t,x(t), \int_0^t g(t,s,x(s))ds\right), \forall t \in [0,+\infty),$$

$$x(0) + H(x) = x_0$$

Note that:

(i) For any $t,s \epsilon [0, +\infty)$ and $x_1, y_1 \in \mathbb{R}$ we have

$|g(t,s,x_1) - g(t,s,y_1)| \leq \frac{24}{7}|sin^3 x_1 - sin^3 y_1|$. (5.3)

Applying mean value theorem to the function $sin^3 x$ on $[x,y]$ with $x < y$ where $x,y \in \mathbb{R}$, there exist $\sigma \in (x,y)$ such that $\frac{sin^3 x - sin^3 y}{x-y} = 3sin^2(\sigma)\cos(\sigma)$.

Therefore, $|sin^3 x - sin^3 y| \leq 3|x - y|$. (5.4)

From (5.3) and (5.4), we have $|g(t,s,x_1) - g(t,s,y_1)| \leq \frac{72}{7}|x_1 - y_1|$.

(ii) Let any $t \epsilon [0, +\infty)$ and $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Then we have

$|f(t,x_1,x_2) - f(t,y_1,y_2)| \leq \left\{\frac{45}{7}|\cos x_1 - \cos y_1| + \frac{5}{7}|\sin 3x_1 - \sin 3y_1|\right\} + |x_2 - y_2| \leq \left\{\frac{45}{7}|x_1 - y_1| + \frac{5}{7}|3x_1 - 3y_1|\right\} + |x_2 - y_2|$

$$\leq \frac{60}{7}\{|x_1 - y_1| + |x_2 - y_2|\}.$$

Next, for any $\epsilon > 9$ we see that $y(t) = \frac{t}{3}$ satisfy the following inequation

$$|\; y'(t) - f\left(t,y(t), \int_0^t g(t,s,y(s))ds\right)|$$

$$= \left|y'^{(t)} - \frac{2}{5} + \frac{45}{7}\cos(y(t)) + \frac{5}{7}\sin(3y(t)) - \frac{24}{7}\int_{\mathbf{0}}^{\boldsymbol{t}} sin^{\mathbf{3}}(y(s))\, ds\right|$$

$$\leq 8.9952 < \epsilon$$

But for any solution $x(t)$ of Eq. (5.1) we have

$|x(t) - y(t)| = |x(t) - \frac{t}{3}| \leq |x(t)| + \frac{t}{3} \to \infty as\; t \to \infty$.Thus, Eq. (5.1) is not Ulam–Hyers stable on $[0, +\infty)$.

6) Conclusions

in this paper, Pachpatte’s inequality is used to established result on the Ulam Hyers stabilities for Volterra integrodifferential equations with nonlocal condition in Banach spaces on infinite interval. Example is given to show the applicability of our obtained result.

7) References

[1] A. Pazy, Semigroup of Linear Operators and Applications to Partial Differential Equations (Springer Verlag, New York, 1983).

[2] Kishor D. Kucche and Pallavi U. Shikhare,Ulam–Hyres stability of integrodifferential equations in banach spaces via Pachpatte's inequality Asian-European Journal of Mathematics Vol. 11, No. 2 (2018) 1850062 (19 pages)

[3] B. Pachpatte, Inequalities for Differential and Integral Equations (Academic Press, New York, 1998).

[4] Byszewski L.; Theorems about the existence and uniquess of solutions of a semilinear evolution nonlocal Cauchy problem, *J. Math. Anal. Appl.*, 162(1991), pg 494-505.

[5] H.L. Tidke and R.T. More ; Existence and uniqueness of mild solutions of nonlinear difference-integrodifferential equation with nonlocal condition, *Journal of Advances in Mathematics*, Vol. 9(2014), No. 3, pg. 2415-2430.

[6] S. M. Ulam, Problems in Modern Mathematics, Chapter 6 (John Wiley and Sons, New York, NY, USA, 1960).

[7] S. M. Ulam, A Collection of Mathematical Problems (Interscience, New York, 1960).

[8] L. P. Castro and R. C. Guerra, Hyers–Ulam–Rassias stability of Volterra integral equation within weighted spaces, Libertas Mathematica 33(2) (2013) 21–35.

[9] D. H. Hyers, On the stability of the linear functional equation, Natl. Acad. Sci. U.S.A. 27 (1941) 222–224.

[10] S. M. Jung, A fixed point approach to the stability of a Volterra integral equations, Fixed Point Theory Appl. 2007 (2007) Article ID 57064, 9 pages.

[11] T. M. Rassias, On the stability of linear mappings in Banach spaces, Proc. Am. Math. Soc. 72 (1978) 297–300.

[12] R.T. More, S.C. Patekar, V.B. Patare, Existence and Uniqueness of Solution of Integrodifferential Equation of Finite Delay in Cone Metric Space, International Journal of Applied Engineering Research Volume 13 no. (23)(2018),pg 16460-16467.

[13]J. Huang, Y. Li: Hyers,Ulam stability of delay differential equations of first order, Math.

Nachr., 289, n. 1, (2016), pg.60-66.

[14] R.T. More , S.C. Patekar,A.P. Nawghare, Study of Ulam Hyers Stability of Integrodifferential Equations with nonlocal Condition in Banach Spaces,J. Math. Comput. Sci. 10 (2020), 236-247.

[15]I. Rus, Ulam stability of ordinary differential equations, Studia.“BABES-BOLYAI”,

Mathematica 54(4) (2009) 125–133.